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Low energy levels and matrix elements in the large-N (small-$\hbar$) asymptotic region

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Abstract. The asymptotic form of the Fröman and Fröman solutions to the Schrödinger equation has been used to calculate the low energy spectra and corresponding matrix elements for many-well potentials. It has been proved that the Bohr-Sommerfeld quantisation condition is also valid in the low energy region, independently for each well. It has also been shown that the corresponding matrix elements are dominated by the energy levels belonging to the same well. The general results have then been used to show that, in N dimensions, with spherically symmetric Hamiltonians, quantum systems approach their classical configurations with constraints when $N\to\infty$.

1. Introduction

Approximate calculations with the help of asymptotic expansions are the common practice in quantum mechanics as well as in quantum field theory. The best known and oldest are the asymptotic expansions in $\hbar$—the Planck constant. Recently, large-N expansions for different quantities (energy, matrix elements, etc) have also been widely used, N being the number of degrees of freedom connected with some kinds of symmetry (O(N), SU(N), etc). Many field-theoretical models are simplified greatly in the limit $N\to\infty$. Results of such calculations are expected to be good approximations for real cases even when N is finite (see, for example, Coleman 1980, Witten 1979). On the other hand, it has been observed that in the large-N limit quantum theories can approach their classical configurations with constraints (Jevicki and Papanicolaou 1980, Jevicki and Levine 1980, Bardakci 1981, Halpern 1981, Yaffe 1982).

The corresponding asymptotic expansions for N (or $\hbar^{-1}$)$\to\infty$ can be obtained by using the Feynman path integral method together with the instanton technique (see, for example, Hikami and Brézin 1979) or with the stationary phase approximation (Dashen *et al* 1974, Rajaraman 1982). The universality of the method allows us to get the asymptotic expansions for the systems with finite degrees of freedom (i.e. in quantum mechanics) as well as in quantum field theory.

However, in the case of quantum mechanics there exists an older alternative method based on the asymptotic solutions to the Schrödinger equation. The method is well known (see, for example, Landau and Lifshitz 1965, Fedoryuk 1983). In some applications it seems to be even simpler than the Feynman path integral method (Koudinov and Smondyrev 1983). Therefore, it is worthwhile to re-examine it by applying it to physically interesting cases.

In the present paper we are going to apply the method to the lower part of the energy spectrum. However, we are faced then with the situation that there are no rules allowing us to find the relevant asymptotic expansions for both low energy levels and matrix elements. Existing applications of the method, although well worked out (see, for example, Landau and Lifshitz 1965, Fröman 1966, Fröman and Fröman 1974a, b, 1977, Berry and Mount 1972, Hioe *et al* 1978), deal mainly with the high energy part of the energy spectrum.

On the other hand, in the known applications of the method to the low energy region the corresponding calculations were performed in a way suitable for the case considered and therefore are difficult for generalisation to an arbitrary case (see, for example, Koudinov and Smondyrev (1983), where the relevant calculations were made in the case of $O(N)$-symmetric anharmonic oscillators regarded in the limit $N \to \infty$).

Therefore, the need arises to develop the corresponding rules for asymptotic calculations in the low energy region in a systematic way. In this paper the relevant rules are formulated for:

(i) quantisation of energy levels; and

(ii) calculations of matrix elements.

To establish the rules, one can proceed along two possible ways.

In the first one, all the relevant calculations are performed using a formal form of the exact solutions to the Schrödinger equation and then taking the corresponding asymptotic limit in the final formulae for energy levels or matrix elements.

The other possibility is to perform all the calculations using the relevant asymptotic representations for the wavefunctions from the very beginning. The final formulae are then readily obtained in their asymptotic forms.

Choosing the first way one has to operate with the special form of the solutions to the Schrödinger equation which allows us to take the desired asymptotic limit in the final formulae in an easy way. The solutions found by Fröman and Fröman (1965) have such desired forms. However, their main disadvantage is a rather complicated analytic structure, which makes it difficult to use them effectively in order to obtain the rules we are looking for.

Contrary to this the asymptotic forms of the Fröman and Fröman solutions are extremely simple and they reveal their usefulness in asymptotic calculations just in the higher regions of the energy spectrum (see, for example, Landau and Lifshitz (1965), Fröman and Fröman (1977) and § 3 of the present paper for a more systematic treatment).

However, when trying to apply the same asymptotic forms of the Fröman and Fröman solutions to the low energy region one meets serious difficulties related to a phenomenon typical for this region. This is a settling down of the low energetic quantum object at the bottom of the quantised potential well if the asymptotic limit is taken.

At the formal level of the asymptotic solutions the phenomenon causes a collapse of two neighbouring classical turning points. Then the size of the classically allowed region defined by the points vanishes asymptotically. As a consequence of that some asymptotic solutions become singular at the bottom of the well. This causes a serious problem in an analytic continuation of the relevant asymptotic solutions through the region. Therefore, direct applications of the asymptotic solutions to the low energy region calls for solving this problem.

It has been shown in this paper that the relevant analytic continuation is possible asymptotically (see § 4 for an introduction to the problem and appendix 5 for its solution). It has made it possible to obtain the following main results:

(i) the proof that the Bohr–Sommerfeld quantum condition also remains unchanged for the low energy part of the energy spectrum (§ 4); and

(ii) the general asymptotic formulae for the low energy levels and the corresponding matrix elements (§§ 4 and 5).

The above general results are then applied to the case of the N-dimensional Schrödinger equation with a potential having $O(N)$ symmetry when the asymptotic limit $N \to +\infty$ is taken (§ 6).

We would like, however, to stress that in this paper we consider only the dominant asymptotic series expansion, i.e. we neglect all possible subdominant (exponentially small) corrections to energies and matrix elements. It is known that such subdominant contributions control the large-order behaviour of the asymptotic expansions of the relevant quantities (Balian *et al* 1979, Hikami and Brézin 1979, Avan 1984, Avan and de Vega 1983, Koudinov and Smondyrev 1983). Therefore, we have to limit our investigations to a few terms of the considered asymptotic expansions. This limitation is not, however, inherent in the method. In fact, it is possible to get the full dominant and subdominant asymptotic structure within the framework of the Fröman and Fröman (1965) solutions to the Schrödinger equation. A discussion of this point is postponed to another paper.

The paper is organised as follows.

In the next section (§ 2) a set of exact solutions to the Schrödinger equation is introduced and its main properties are discussed. The solutions have the forms as found by Fröman and Fröman (1965). The reasons for performing such an extended discussion of these solutions (despite the fact that there exists the original literature of the subject (see for example Fröman and Fröman 1965, Dammert and Fröman 1980, Dammert 1983, 1986)) is the following.

In applying the solutions to physical problems we prefer to adopt a technique different from the original F-matrix approach of Fröman and Fröman (1965). We rather follow Fedoryuk's (1983) approach using well defined and finite systems of the exact solutions to the Schrödinger equation having the Fröman and Fröman form. In what follows the chosen solutions shall be called the fundamental solutions (§ 2.1).

The following properties of the fundamental solutions are their main advantages in applications.

(i) Each fundamental solution has a well defined asymptotic representation which is valid in a so-called canonical domain of the relevant fundamental solution. The canonical domains and their properties are discussed in § 2.2.

(ii) The set of asymptotic solutions can be used in the same way as the exact fundamental solutions themselves, preserving some of the important properties of the latter in simplified forms. These asymptotic representations and their properties are discussed in § 2.3.

(ii) The finite sets of fundamental solutions to the Schrödinger equation are sufficient to solve each global one-dimensional quantum mechanical problem (or many dimensional reduced to one) such as, for example, the quantisation of energy levels, calculations of matrix elements, etc. This is why they are called the fundamental solutions.

(iv) The property of the fundamental solutions formulated in the previous item is also shared by the sets of corresponding asymptotic representations of the fundamental solutions if the relevant problems are to be solved asymptotically in the region of high energy levels.

In § 3 the Bohr–Sommerfeld quantum conditions are rederived for the high energy levels in the case of many-well potentials. The aim of the section is to show how effective is the direct use of the asymptotic solutions in the high energy region as well as to get necessary experience for a subsequent discussion in the next section.

In § 4 we consider the asymptotics of the low energy part of the spectrum. We show that the corresponding quantisation procedure can be performed and that emerging quantisation conditions are the same as for the high energy levels. Using the conditions some general expressions for the first three terms of the low energy level asymptotic series are obtained.

In § 5 the asymptotic calculations of the relevant low energy matrix elements are performed for the many-well potential. A general result of these calculations is rather simple: the asymptotically dominant part of the relevant matrix diagonalises and each diagonal matrix element is dominated asymptotically by the classical value of the considered quantity at the bottom of the quantised well.

The general results of §§ 4 and 5 are then applied in § 6 where the low-energy spectrum and the matrix elements for the N-dimensional Schrödinger equation with $O(N)$ symmetry are investigated. We find that the quantum mechanical system with $O(N)$ symmetry approaches a classical configuration with constraints when $N\to\infty$. The results of our calculations are in full agreement with those obtained by Jevicki and Papanicolaou (1980), Halpern (1981), Bardakci (1981) and Yaffe (1982).

The results obtained in the paper are summarised in § 7.

For clarity, many detailed calculations, proofs, formulae etc have been presented in appendices.

2. Fundamental solutions to the Schrödinger equation and their asymptotic forms.

2.1. Fundamental solutions to the Schrödinger equation

Let us assume for simplicity that the potential $U(x, \lambda)$ is a polynomial function of x with coefficients being meromorphic functions of λ. $U(x, \lambda)$ is real for real x and λ. We also assume that the spectrum is purely discrete, i.e. $U(x, \lambda)\to+\infty$ for $|x|\to+\infty$. The Schrödinger equation:

$$[(-\hbar^2/2m)(\mathrm{d}^2/\mathrm{d}x^2+U(x, \lambda)]\psi(x)=E\psi(x)$$

can be written in the form:

$$\psi''(x)-\lambda^2 q(x, E, \lambda)\psi(x)=0 \tag{2.1}$$

where $q(x, E, \lambda)=2m[U(x, \lambda)-E]/(\lambda\hbar)^2$.

After making the substitution:

$$\psi_\sigma(x)=q^{-1/4}(x)\exp[\sigma S(x_0, x)]\tilde{\psi}_\sigma(x)$$

$$S(x_0, x)=\lambda\int_{x_0}^{x} q^{1/2}(y)\,\mathrm{d}y \qquad \sigma=\pm 1 \tag{2.2}$$

$$q(x_0)=0$$

we obtain an appropriate equation determining $\psi_\sigma(x)$. Solving by iteration we get

$$\tilde{\psi}_\sigma(x) = 1 + \sum_{n\geq 1} (\sigma/\lambda)^n \int_{\gamma^\sigma(x)} dy_1 \int_{\gamma^\sigma(y_1)} dy_2 \ldots \int_{\gamma^\sigma(y_{n-1})} dy_n \omega(y_1) \ldots \omega(y_n)$$
$$\times \{1 - \exp[2\sigma S(x, y_1)]\}$$
$$\times \{1 - \exp[2\sigma S(y_1, y_2)]\} \ldots \{1 - \exp[2\sigma S(y_{n-1}, y_n)]\} \qquad (2.3)$$

where

$$\omega(y) = \tfrac{1}{8}[q''(y)/q^{3/2}(y) - \tfrac{5}{4}(q'(y))^2/q^{5/2}(y)] \qquad (2.4)$$

and where the dependence of the relevant functions (ψ, q, ω, etc) on the parameters λ, $\hbar$, E is not explicitly indicated.

Zeros of $q(x)$ are called turning points. They are simultaneously the singular points of $\omega(x)$.

In the formula (2.3) the integration path $\gamma^\sigma(y_k)$ starts from infinity and runs to y_k. Besides, Re $S(x_0, y_{k+1}) \to -\sigma\infty$ when $y_{k+1} \to \infty$ along the path.

The series in (2.3) is uniformly convergent provided that

$$\liminf_{\gamma^\sigma(x)} C_{\gamma^\sigma}(x) \int_{\gamma^\sigma(x)} |\omega(y)\, dy| < +\infty \qquad (2.5)$$

where the limit is taken for the set of all the paths $\gamma^\sigma(x)$ and $C_{\gamma^\sigma}(x)$ is defined by

$$C_{\gamma^\sigma}(x) = \max_{y,z \in \gamma^\sigma(x)} |1 - \exp[2\sigma S(y, z)]| \qquad (2.6)$$

where the points y, z are ordered on $\gamma^\sigma(x)$, i.e. y lies between x and z on $\gamma^\sigma(x)$.

A convenient way to describe the main properties of the representations (2.2) and (2.3) is to draw a so-called Stokes graph (see figure 1). Such a graph consists of Stokes lines given by the equation Re $S(x_0, x) = 0$ for all roots x_0 of $q(x)$.

Each system of Stokes lines divides the whole complex x plane into a set of disjoint pieces. The solutions of the form given by (2.2) and (2.3) can be constructed in each piece that contains $+\infty$ or $-\infty$ of Re $S(x_0, x)$.

Each such piece D which does not contain any Stokes line (i.e. the Stokes lines form its boundary ∂D) and for which $\text{Re}[\sigma S(x_0, x)] < 0$ with $x_0 \in \partial D$ we shall call a sector.

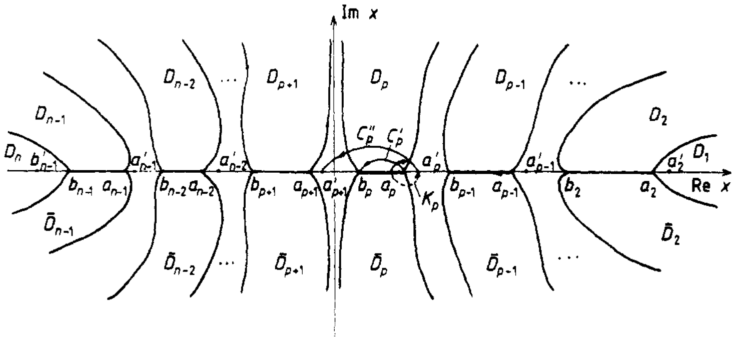

Figure 1. The Stokes graph for a general potential.

In each sector D the solution (2.2) is uniquely determined (up to a constant) by the condition of vanishing as $x \to \infty$ in this sector. The solution (2.2) defined originally in the sector D can be further continued analytically to any point of the Stokes graph (which is not a turning point of the graph) using the formulae (2.2)-(2.4). It increases infinitely when continuing to any other sector of the graph if $x \to \infty$ in this sector.

Therefore, each particular Stokes graph defines its own set of solutions having the form (2.2). We shall call each solution entering the set a fundamental solution to the Schrödinger equation.

Let us note further that any two solutions defined in different sectors of the Stokes graph are linearly independent. This fact is the obvious consequence of their asymptotic behaviour described above.

On the other hand, any one of the fundamental solutions can be expressed as a linear combination of another two linearly independent fundamental solutions. The coefficients of such a linear combination can be calculated directly by analytic continuation of the solutions to the corresponding sectors of the Stokes graph keeping their form (2.2). Writing

$$\psi_1(x) = \alpha_{i/j\to k}\psi_j(x) + \alpha_{i/k\to j}\psi_k(x) \tag{2.7}$$

we have

$$\begin{aligned} \alpha_{i/j\to k} &= \lim_{x\to\infty_k} [\psi_i(x)/\psi_j(x)] \\ \alpha_{i/k\to j} &= \lim_{x\to\infty_j} [\psi_i(x)/\psi_k(x)] \end{aligned} \tag{2.8}$$

where ∞_k, ∞_j are the infinity points at the sectors D_k, D_j, respectively.

2.2. Canonical points and canonical domains of the fundamental solutions

For a discussion of asymptotic properties of the fundamental solutions the notions of canonical points and of canonical domains seem to be extremely useful (Fedoryuk 1983).

Let D be the sector where the fundamental solution ψ_D is defined.

A point x is a canonical point for $\psi_D(x)$ if a path $\gamma_D(x)$ exists for which each ordered pair (y, z) of the points y and z lying on the path (with y between x and z) fulfils:

$$\mathrm{Re}[\sigma S(y, z)] \leqslant 0. \tag{2.9}$$

A path $\gamma_D(x)$ with the property (2.9) is called a canonical path.

It follows from (2.9) that for the canonical points of $\psi_D(x)$ the coefficient $C_{\gamma_D}(x)$ defined in (2.6) cannot be greater than 2.

The set K_D of all canonical points of ψ_D is called the canonical domain of ψ_D. Of course, $D \subset K_D$.

We shall call the domains K_{D_1} and K_{D_2} communicated canonical domains if D_1, $D_2 \subset K_{D_1} \cap K_{D_2} \neq \emptyset$.

Let K_{D_i} $i = 1, 2, 3$, be the pairwise communicating canonical domains. It follows then that for the corresponding solutions ψ_{D_i}, $i = 1, 2, 3$, the coefficients $\alpha_{D_i/D_j\to D_k}$, $\alpha_{D_i/D_k\to D_j}, \ldots$, etc, $i, j, k = 1, 2, 3$, of the corresponding linear combination (2.7) can be calculated by (2.8) keeping x running to the infinities ∞_{D_j}, $\infty_{D_k}, \ldots$, etc, along the corresponding canonical paths $\gamma_{D_i}(\infty_j)$, $\gamma_{D_i}(\infty_k), \ldots$, etc. Such α coefficients we shall call canonical coefficients.

If K_{D_0} and $K_{D_{n+1}}$ are not communicating canonical domains then it is always possible to find a sequence $K_{D_1}, K_{D_2}, \ldots, K_{D_n}$ of the canonical domains such that in each of the triads $(K_{D_p}, K_{D_{p+1}}, K_{D_{p+2}})$, $p = 0, 1, \ldots, n-1$, the domains are pairwise communicating canonical domains.

It follows, therefore, that in each linear relation (2.7) its coefficients are either the canonical coefficients or can be expressed as a function of such coefficients.

The last property is the key one for asymptotic calculations of energy levels and matrix elements when asymptotic forms of the fundamental solutions are used directly instead of the solutions themselves.

Finally, let us note that, as follows from (2.2), in order to get α in (2.8) we have to calculate $\tilde{\psi}_i(\infty_j) \equiv \tilde{\psi}_{i\to j}$, $\tilde{\psi}_{i\to k}$, etc, defined in (2.3). An important property of these quantities is the following one (see appendix 1):

$$\begin{aligned} &\tilde{\psi}_{i\to j} = \tilde{\psi}_{j\to i} \\ &\tilde{\psi}_{i\to k} = \tilde{\psi}_{k\to i} \\ &\text{etc.} \end{aligned} \tag{2.10}$$

2.3. λ asymptotic of the fundamental solutions

Each fundamental solution has a well defined asymptotic behaviour when $\lambda \to +\infty$ in the sector where it is constructed under the assumption that the corresponding Stokes graph does not change 'essentially' when the limit is taken. This can be expressed more precisely as follows (Fedoryuk 1983).

Let $\lambda > \lambda_D \gg 1$, and $z_1(\lambda), \ldots, z_{2n}(\lambda)$ be the (simple) roots of $q(x, E, \lambda)$. Let us assume further that for $\lambda \to +\infty$ and any $k = 1, \ldots, 2n$ finite limits exist: $\lim z_k(\lambda) = z_k$. Let also $q(x, E, \lambda)$ have the following asymptotic behaviour when $\lambda \to +\infty$:

$$q(x, E, \lambda) \sim q_{\mathrm{AS}}(x, E, \lambda) = q_0(x) - 2E + \sum_{n\geq 1} q_n(x)\lambda^{-n}. \tag{2.11}$$

If for any $k = 1, \ldots, 2n$ and $\lambda > \lambda_D$ there are pairwise disjoint domains Ω_k of the x plane, such that $z_k(\lambda)$, $z_k \in \Omega_k$ then each fundamental solution ψ_p has in the sector $D_p, p = 1, 2, \bar{2}, \ldots, r-1, \overline{r-1}, r$, where it is defined, the following asymptotic expansion, for $\lambda \to +\infty$:

$$\psi_{p,\mathrm{AS}} = q_{\mathrm{AS}}^{-1/4}(x, E, \lambda) \exp\left(\lambda\sigma \int_{z_k}^{x} q_{\mathrm{AS}}^{1/2}\, \mathrm{d}y + \int_{\infty_p}^{x} \chi(\sigma, y, E, \lambda)\, \mathrm{d}y\right) \tag{2.12}$$

where the asymptotic series $\chi(\sigma, y, E, \lambda)$ is constructed in appendix 2.

The asymptotic form (2.12) for the fundamental solution ψ_D can be continued analytically to each point of its canonical domain K_D. On the other hand, K_D is the *maximal* domain where the representation (2.12) can exist. However, the asymptotic form $\psi_{D,\mathrm{AS}}$ of ψ_D can be found outside its canonical domain K_D by the same analytic continuation procedure which has been described for ψ_D itself at the end of § 2.2. Besides, let us note that there is no other way of analytic continuation of (2.12) outside K_D. In particular, continuing $\psi_{D,\mathrm{AS}}$ in this way, one can use for this goal the relations (2.7) and (2.8). This will be demonstrated explicitly in the next section.

Let us note further that the relations (2.10) remain valid also asymptotically, being expressed more explicitly as conditions on the functions $\chi(\sigma, x, E, \lambda)$ in the following way.

Let us write $\chi(\sigma)$ $(\equiv \chi(\sigma, x, E, \lambda))$ as

$$\begin{aligned} \chi^+(\sigma) &= \tfrac{1}{2}[\chi(\sigma) + \chi(-\sigma)] \\ \chi^-(\sigma) &= \tfrac{1}{2}[\chi(\sigma) + \chi(-\sigma)] \end{aligned} \tag{2.13}$$

where χ^+ and χ^- stand for the even and odd parts of $\chi(\sigma)$, respectively (i.e. $\chi^\pm(\sigma) = \pm\chi(-\sigma)$). It means, of course, that χ^+ does not depend on σ, i.e. $\chi^+(\sigma) \equiv \chi^+(x, E, \lambda)$ and $\chi^-(\sigma) = \sigma\chi^-(x, E, \lambda)$.

Let us note further that the sectors D_i, D_j in (2.10) are such that $\sigma_i = -\sigma_j$ or if $\sigma_i = \sigma_j$ then the continuation of ψ_i to D_j has to cross the cut where $q^{1/2}$ changes its sign. Therefore, effectively $\sigma_i = -\sigma_j$ in any case. On the other hand we have the following asymptotic representation for $\tilde{\psi}_i(x)$:

$$\tilde{\psi}_{i,\mathrm{AS}}(x) = \exp\left(\int_{\infty_i}^{x} \chi(\sigma_i, y, E, \lambda)\, \mathrm{d}y\right) \tag{2.14}$$

which comes out from (2.12). Using (2.13) we get from (2.10):

$$\int_{\infty_i}^{\infty_j} \chi^+(y, E, \lambda)\, \mathrm{d}y \equiv 0 \tag{2.15}$$

with no condition on χ^-. It follows further from (2.15) that the integral

$$\int_{\infty_p}^{x} \chi^+(y, E, \lambda)\, \mathrm{d}y \tag{2.16}$$

does not depend on what sector D_p, $p = 1, 2, \bar{2}, \ldots, r-1, \overline{r-1}, r$, the infinity point ∞_p is taken in.

Finally, the following statement expresses the importance of the asymptotic representations (2.12) for asymptotic calculations.

The set of asymptotic forms (2.12) of the fundamental solutions is sufficient for solving asymptotically each global (one-dimensional) quantum mechanical problem in the region of high energies.

The above statement is the asymptotic analogue of the basic property of the set of the fundamental solutions formulated in the introduction.

3. High energy levels and matrix elements in the asymptotic region $\lambda \to +\infty$

The assumptions under which the asymptotic representation (2.12) has been written lead to the conclusion that (2.12) can be applied directly to the higher lying levels of the energy spectrum. This limitation follows as a consequence of a *finite* separation of the turning points z_k, $k = 1, \ldots, 2n$, in the limit $\lambda \to +\infty$, since then, as follows from (2.2) and (2.12), for each pair z_k, z_j, $k \neq j$, $k, j = 1, \ldots, 2n$, of the turning points we have

$$\lim_{\lambda \to +\infty} S(z_k, z_j) = \infty. \tag{3.1}$$

In particular, if z_k, z_j are a pair of the real classical turning points related to some potential well, then (3.1) means that the momentum of a particle bounded in the well increases infinitely in the limit $\lambda \to +\infty$.

However, since the limiting Stokes graph remains almost unchanged in comparison with the original one then the asymptotic representations (2.12) can be applied using the same rules in relevant calculations which are proper for the fundamental solutions themselves.

On the other hand, if $S(z_k(\lambda), z_j(\lambda))$ is to remain finite in the limit $\lambda \to +\infty$ for some pair $z_k(\lambda)$, $z_j(\lambda)$ of turning points then the points have to collapse in this limit. That is, we should have

$$\lim_{\lambda \to +\infty} [z_k(\lambda) - z_j(\lambda)] = 0. \tag{3.2}$$

The relation (3.2) means that the limiting Stokes graph has to change dramatically in such a case. Further, although the asymptotic representations (2.12) remain valid in each sector of the limiting Stokes graph some paths of analytic continuations previously possible are now blocked. Therefore, it is also unclear how to use then the representations (2.12) in an effective way.

Postponing the discussion of this problem to the next section (§ 4) we shall first show how the representations (2.12) work when used to quantise high energy levels. Due to that we shall also get some insight into what will happen when the collapse (3.2) takes place.

3.1. *The Bohr-Sommerfeld quantisation condition*

Let figure 1 represent the asymptotic form of the Stokes graph for energy E when $\lambda \to +\infty$. Then (2.12) defines the asymptotic forms of the solutions (2.3) in the sectors $D_1, D_2, \bar{D}_2, \ldots, D_{n-1}, \bar{D}_{n-1}, D_n$, with $\sigma_k = (-1)^k$, where k is the sector number. If $\psi^{\mathrm{AS}}(x, E, \lambda)$ is an asymptotic form of the (normed) wavefunction $\psi(x, E, \lambda)$ with energy E then we have

$$\psi^{\mathrm{AS}}(x, E, \lambda) = \begin{cases} C_1 \psi_{1,\mathrm{AS}}(x, E, \lambda) & x \in D_1 \\ C_n \psi_{n,\mathrm{AS}}(x, E, \lambda) & x \in D_n \end{cases}. \tag{3.3}$$

In order to get the asymptotic forms of ψ in the remaining sectors $D_2, \bar{D}_2, \ldots, D_{n-1}, \bar{D}_{n-1}$, we have to continue analytically the solutions $\psi_{1,\mathrm{AS}}$ and $\psi_{n,\mathrm{AS}}$ to these sectors. It can be done with the help of the asymptotic solutions defined in these sectors together with formulae (2.7) and (2.8) (see appendix 3). Finally, $\psi_{1,\mathrm{AS}}$ and $\psi_{n,\mathrm{AS}}$ continued to the sectors D_k and $\bar{D}_k$ are

$$\begin{aligned} \psi_{1,\mathrm{AS}} &= \sigma_k \psi_{k,\mathrm{AS}} + \bar{\sigma}_k \psi_{\bar{k},\mathrm{AS}} \\ \psi_{n,\mathrm{AS}} &= \omega_k \psi_{k,\mathrm{AS}} + \bar{\omega}_k \psi_{\bar{k},\mathrm{AS}} \end{aligned} \tag{3.4}$$

where σ_k and ω_k are given in appendix 3. If we now match $\psi_{1,\mathrm{AS}}$ and $\psi_{n,\mathrm{AS}}$ at the sector D_k we get the following asymptotic quantisation condition for the energy:

$$\prod_{q=2}^{n-1} (1 + e_q) = 0 \tag{3.5}$$

where

$$e_q = \exp\left((-1)^{q+1} \oint_{C_q} (\lambda q_{\mathrm{AS}}^{-1/2} + \chi^-) \right) \qquad q = 2, \ldots, n-1.$$

Suppose that (3.5) is satisfied due to $e_r = -1$. It is then seen from formulae (A3.2) and (A3.5) that $\psi_{1,\mathrm{AS}}$ and $\psi_{n,\mathrm{AS}}$ can both be continued at most to sector D_r. Therefore, the constant $C = C_n / C_1$ which relates both the considered solutions can be determined by matching $\psi_{1,\mathrm{AS}}$ and $\psi_{n,\mathrm{AS}}$ exactly in this sector. Using again the formulae (A3.2) and (A3.5) this gives

$$C=\alpha_r/\omega_r=\mathrm{i}\exp\left[-\int_{\infty_1}^{a_2'}\chi^-+\sum_{p=2}^{r-1}(-1)^p\left(\int_{C_p'}\lambda q_{\mathrm{AS}}^{1/2}+\int_{C_p''}\chi^--\int_{K_p}\chi^-\right)\right.$$
$$\left.-\sum_{p=r}^{n-2}(-1)^p\left(\int_{C_p'}\lambda q_{\mathrm{AS}}^{1/2}+\int_{C_p''}\chi^--\int_{K_p}\chi^-\right)+(-1)^n\int_{\infty_n}^{b_{n-1}'}\chi^-\right]$$
$$\times\prod_{p=2}^{r-1}(1+e_p)\left(\prod_{p=r+1}^{n-1}(1+e_p)\right)^{-1}. \tag{3.6}$$

The quantisation condition (3.5) follows as a result of the reality of C, i.e. $C=\bar{C}$. It is thus seen from (3.5) that the energy is quantised asymptotically in independent ways for each well of the potential. For the rth well the quantisation condition is

$$\mathrm{i}(-1)^r\oint_{C_r}(\lambda q_{\mathrm{AS}}^{1/2}+\chi^-)=(2m+1)\pi \qquad m=0,1,\ldots. \tag{3.7}$$

Let us now note that application of the condition (3.7) strongly depends on our assumption about the 'stability' of the Stokes graph (see figure 1) in the limit $\lambda\to+\infty$. In particular this assumption means that all the distances between the points a_k, b_k and b_k, a_{k+1}, $k=2,\ldots,n-1$, in figure 1 remain finite when $\lambda\to+\infty$. It follows then immediately that for the energies $E_{r,m}$ fulfilling the conditions (3.7) for some r and m we have $E_{r,m}\to$ constant ($>U(x_r)$, with x_r the minimum of the rth well). Therefore, for the quantum number m in (3.7) we have $m\sim\lambda$ in the limit $\lambda\to+\infty$ (since $\chi^-\sim\lambda^{-1}$ in this limit). In this way the stability assumption provides us with the usual condition for applications of the Bohr-Sommerfeld formula (3.7), i.e. we should apply them to the high energy levels. It follows therefore that the region of the low energies has to be considered separately.

3.2. λ asymptotics for matrix elements

Let us discuss briefly how to use the asymptotic forms (2.12) to calculate the $\lambda\to+\infty$ asymptotics for different matrix elements. We are interested in the matrix elements of the functions $f(x,\mathrm{d}/\mathrm{d}x)$ that are polynomials in $\mathrm{d}/\mathrm{d}x$ with holomorphic (or meromorphic) coefficients $V_k(x)$, finite for real and finite x. The relevant matrix elements are then

$$f_{EE'}(\lambda)=\int_{-\infty}^{+\infty}\psi(x,E,\lambda)f(x,\mathrm{d}/\mathrm{d}x)\psi(x,E',\lambda)\,\mathrm{d}x. \tag{3.8}$$

However, to get the $\lambda\to+\infty$ asymptotics of (3.8) it is insufficient to substitute ψ in the left-hand side of (3.8) by their asymptotic form (2.12). It is also necessary to distort the integration path in (3.8) in such a way so as to pass through extrema of $\mathrm{Re}(\int_{z_k}^x q_{\mathrm{AS}}^{1/2}+\int_{z_k}^x q_{\mathrm{AS}}'^{1/2})$, which is the usual condition for using the saddle-point method. This is the way of finding the asymptotic expressions for the matrix elements (3.8) which has been described by Landau and Lifshitz (1965) (see also Migdal 1975).

There also exists an alternative approach, developed by Fröman and Fröman (1977) (see also Streszewski and Jedrzejek (1988) for the Feynman path version of this method). In what follows, however, we shall prefer to use directly the saddle-point method to investigate the matrix elements (3.8) because of the regular behaviour of the low energetic asymptotic wavefunctions at the critical points lying on the real axis (see appendix 4).

4. The low energy level λ asymptotics

4.1. *Are the Bohr–Sommerfeld conditions valid for the low energy levels?*

As we have noticed earlier, the case of low energy levels has to be considered separately. At first glance, what happens exactly in that case seems to be inferred from (3.7) where for *fixed m* and $\lambda \to +\infty$ we expect to get from

$$\oint_{C_r} [q_0(x) - 2E_m^{(0)}]^{1/2} = 0 \tag{4.1}$$

where $E_m^{(0)} = \lim_{\lambda\to+\infty} E_\infty(\lambda)$. However, there are at least two objections against such a procedure. First, as we have seen, to obtain the quantisation condition (3.7) it was necessary to continue first the solutions $\psi_{1,\mathrm{AS}}$ and $\psi_{n,\mathrm{AS}}$ to the sector r with the help of the formulae (3.4). On the other hand, the coefficients σ_r and ω_r that define the continuations depend on the factors $\exp[\Sigma_{p=n}^{r-1} (-1)^{p+1} \int_{K_p} \chi^-]$ and $\exp[\Sigma_{p=2}^{r+1} (-1)^p \int_{K_p} \chi^-]$, respectively, which can spoil the whole procedure in the case of low energies. The crucial point is that the integration contours K_p in these factors are pinched by the points b_p, a_p, respectively, in the limit $\lambda \to +\infty$. Therefore, one can doubt seriously whether the corresponding integrations remain finite when the limit is taken because the integrand χ^- is singular at each minimum $x_p = \lim_{\lambda\to+\infty} b_p = \lim_{\lambda\to+\infty} a_p$.

Secondly, if the derivatives $q^{(i)}(x_r)$ vanish for $i = 1, \ldots, 2s-1$, $s \neq 1$ (as in the case of the quartic anharmonic potential $U(x) = x^4$), then not only the points a_r and b_r collapse at x_r when $\lambda \to +\infty$ but also $2s-2$ other (complex) zeros of $q_0(x)$ which lie outside the contour C_r in (3.7). In such a case the situation becomes 'pathological' for the quantisation condition (3.7) because the contour C_r is pinched by these $2s-2$ collapsing complex zeros against the zeros at a_r and b_r. Such a pinching is strictly connected with an asymptotic dependence of energy E on λ: the dependence is 'analytical' when $s = 1$ and singular when $s > 1$ with a branch point at $\lambda^{-1} = 0$. Therefore, in further considerations we limit ourselves to the simpler cases with two collapsing real zeros (i.e. to the cases with $s = 1$). For such cases it is shown in appendix 5 that the 'danger' integrations along the contours K_p remains finite when $\lambda \to +\infty$. Consequently, the asymptotic quantisation condition (3.7) remains valid also for the low energy levels, the equation (4.1) is correct and we can continue its analysis.

4.2. *The low energy levels in the limit $\lambda \to +\infty$*

First, (4.1) means that there is *no* cut between the points a_r, b_r. This is possible only when the difference $q_0(x) - 2E_m^{(0)}$ is proportional to $(x - x_r)^2$, i.e. when $E_m^{(0)} = q_0(x_r)$, $q_0'(x_r) = 0$ and $q_0''(x_r) \neq 0$. Of course, it means that our quantum object settles down at the bottom of the rth well when $\lambda \to +\infty$. For the Stokes graph of figure 1 it results in collapse of the points a_r, b_r into the point x_r.

To handle these cases with the help of the quantisation rules (3.7) it is merely necessary to fulfill the following conditions.

(i) To write E in (3.7) as a series:

$$E_{r,m}=\sum_{k\geq 0} E^{(k)}_{r,m}\lambda^{-k}. \tag{4.2}$$

(ii) To expand the integrand $\lambda q_{AS}^{1/2}+\chi^-$ into power series in λ:

$$\lambda q_{AS}^{1/2}+\chi^-=\lambda\bar{q}_0^{1/2}+\tfrac{1}{2}\bar{q}_1\bar{q}_0^{-1/2}+\lambda^{-1}(\tfrac{1}{2}\bar{q}_2\bar{q}_0^{-1/2}-\tfrac{1}{8}\bar{q}_1\bar{q}_0^{-3/2}+\chi^-)+\ldots \tag{4.3}$$

where $\bar{q}_p\equiv q_p-2E^{(p)}_{r,m}$, $p=0,1,\ldots$.

(iii) To substitute each $\chi_k(x;q_0-2E,\ldots,q_k)$ by

$$\chi_k(x;q_0-2E^{(0)}_{r,m},\ldots,q_k-2E^{(k)}_{r,m}) \qquad k=2,\ldots.$$

(iv) To perform the integrations in (3.7) term by term.

With the above prescriptions we obtain the following results for the first three terms in (4.2):

$$i(-1)^r\oint_{C_r}(q_0-2E^{(0)}_{r,m})^{1/2}\,dy=0$$

and

$$E^{(0)}_{r,m}=\tfrac{1}{2}q_0(x_r)$$

$$i(-1)^r\oint_{C_r}\tfrac{1}{2}(q_1-2E^{(1)}_{r,m})(q_0-2E^{(0)}_{r,m})^{-1/2}\,dy=(2m+1)\pi$$

and

$$E^{(1)}_{r,m}=(m+\tfrac{1}{2})[2q_0''(x_r)]^{1/2}+\tfrac{1}{2}q_1(x_r)$$

$$i(-1)^r\oint_{C_r}[\tfrac{1}{2}\bar{q}_2\bar{q}_0^{-1/2}-\tfrac{1}{8}\bar{q}_1\bar{q}_0^{-3/2}+\chi^-]\,dy=0$$

and

$$E^{(2)}_{r,m}=\tfrac{1}{4}\{\tfrac{1}{4}[(m+\tfrac{1}{2})^2+\tfrac{1}{4}]q''_{0,r}q^{(4)}_{0,r}-\tfrac{5}{12}[(m+\tfrac{1}{2})^2+\tfrac{7}{60}](q^{(3)}_{0,r})^2+(m+\tfrac{1}{2})(2q''_{0,r})^{1/2}$$
$$\times(q''_{0,r}q''_{1,r}-q^{(3)}_{0,r}q'_{1,r})-q''_{0,r}(q'_{1,r})^2\}(q''_{0,r})^{-2}+\tfrac{1}{2}q_{2,r} \qquad m=0,1,\ldots \tag{4.4}$$

where $q_{n,r}=q_n(x_r)$, $q'_{n,r}=q'_n(x_r),\ldots$, etc, $n=0,1,\ldots$.

5. The low energy matrix element λ asymptotics

Let us now discuss the possibility of asymptotic calculations of the matrix elements of the type (3.8) in the considered case. The general method described by Landau and Lifshitz (1965) can also be applied now. But it can be simplified greatly because the relevant saddle points lie on the real axis in the considered case and therefore the deformation of the integration contour in (3.8) is no longer necessary.

However, to handle properly all the relevant cases let us list first the possible variety of the matrix elements (3.8) we should take into account. In our further considerations we shall limit ourselves (without loss of generality) to the cases when the low energy levels are quantised at most in two wells, say k and p (with $k>p$), with the help of the rules (i)–(iv) stated above. We assume also that the bottom of the kth well lies *below* the pth one and that $q_0(x_p)=q_0'(x_p)=0$ and $q_0''(x_p)=1$, where x_p stands for the position of the bottom of the pth well. The bottom of all other wells lie above the x axis. Then we can specify the following cases of the matrix elements (3.8) in which there is no need to deform the integration path off the real axis.

(i) Both the energies E and E' are low and belong to the kth well energy spectrum and

$$q_0(x_k)<0 \tag{5.1}$$

where x_k is the position of the kth well bottom.

(ii) Both the energies E and E' are low and belong to the spectrum of the pth well and

(a) $q_0(x_k)<0$,

(b) $q_0(x_k)=0,\ q_0(x_k)=1,\ldots,\ q_0^{(r)}(x_k)=q_0^{(r)}(x_p),\ q_1(x_k)=q_1(x_p),\ldots,$

$$q_1^{(r-1)}(x_k)=q_1^{(r-1)}(x_p),\ldots,q_r(x_k)=q_r(x_p) \tag{5.2}$$

(c) $\qquad q_{AS}(x)=q_{AS}(-x)$, (asymptotically symmetric potential)

(iii) E belongs to the pth well low energy spectrum and E' to the kth one and

$$q_0(x_k)=0,\ q_0(x_k)=1,\ldots,\ q_0^{(r)}(x_k)=q_0^{(r)}(x_p),$$
$$q_1(x_k)=q_1(x_p),\ldots,q_1^{(r-1)}(x_k)=q_1^{(r-1)}(x_p),\ldots,q_r(x_k)=q_r(x_p). \tag{5.3}$$

For the remaining possible combinations of the energies E and E' (also including high lying levels) one needs to use the general Landau and Lifshitz prescription.

Let us now consider successively all the cases listed above. However, in order not to complicate excessively our further investigations we suppose that the function f in the matrix elements (3.8) depends only on x (i.e. f is independent of $\mathrm{d}/\mathrm{d}x$).

Case 1. In fact, this is the case of a one-well potential (see figure 2). The general formulae (3.4) for $\psi_{3,\mathrm{AS}}$ (since $n=3$) and $\psi_{1,\mathrm{AS}}$ are significantly simplified in this case. Both $\psi_{3,\mathrm{AS}}$ and $\psi_{1,\mathrm{AS}}$ are holomorphic at the point $x=x_k$ (see appendix 4). The matrix elements (3.8) can be calculated without deformation of the integration path. So, expanding $f(x)$ in power series around the point x_k and using the asymptotic formula of appendix 6, we get

$$\begin{aligned} f_{m,n}&=\left((-1)^{m+n}\int_{-\infty}^{x_k}\psi_{3,\mathrm{AS}}^{(m)}f(x)\psi_{3,\mathrm{AS}}^{(n)}\,\mathrm{d}x+\int_{x_k}^{+\infty}\psi_{1,\mathrm{AS}}^{(m)}f(x)\psi_{1,\mathrm{AS}}^{(n)}\,\mathrm{d}x\right)\\ &\qquad\times\left(\int_{-\infty}^{+\infty}(\psi_{1,\mathrm{AS}}^{(m)})^2\int_{-\infty}^{+\infty}(\psi_{1,\mathrm{AS}}^{(n)})^2\right)^{-1/2}\\ &=f(x_k)\delta_{mn}+\mathrm{O}(\lambda^{-1}). \end{aligned} \tag{5.4}$$

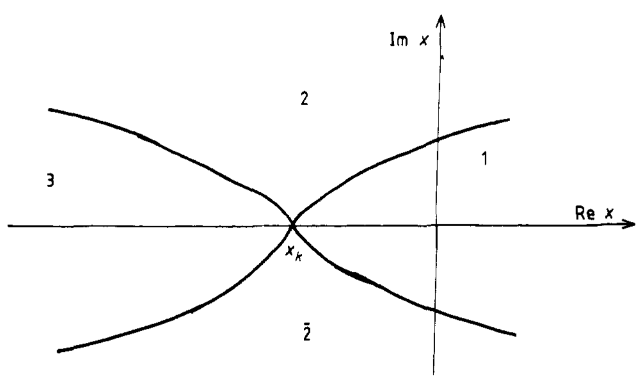

Figure 2. The Stokes graph for case 1 of § 5.

To obtain the final result in (5.4) we have made use of the orthogonality relations for $\psi_{1,\mathrm{AS}}^{(m)}$ and $\psi_{1,\mathrm{AS}}^{(n)}$ and of the fact that the rth term of the power series expansion for $f(x)$ (proportional to $(x-x_k)^r$) lowers by r the leading power of λ in the corresponding asymptotic series.

Let us note, however, that with the help of our method of calculation it would be extremely difficult to find explicitly the general form of the coefficients of the asymptotic series in (5.4), except for the first few. The method of Balian *et al* (1979) seems to be much more effective for finding the large order behaviour of these coefficients with λ (see, for example, Koudinov and Smondyrev 1983).

Case 2a. This is the case of the double-well potential (see figure 3). It is necessary to use different asymptotic representations for the wavefunctions ψ_m and ψ_n on different parts of the integration path K. Let us choose them as follows:

$$\begin{array}{ll} C\psi_{4,\mathrm{AS}}(x) & \text{for the part } K_1, \text{ of } K \\ C(\alpha\psi_{3,\mathrm{AS}}(x)+\bar{\alpha}\psi_{\bar{3},\mathrm{AS}}(x)) & \text{for the parts } K_2, K_3, K_4 \text{ and } K_5 \\ \psi_{1,\mathrm{AS}}(x) & \text{for the part } K_6 \text{ of } K. \end{array} \tag{5.5}$$

The deformation of K from the x axis, as seen in figure 3, is necessary since the points b_1, a_2 are singular for all the solutions $\psi_{i,\mathrm{AS}}$, $i=1,3,\bar{3},4$. The constant C in (5.5) is defined by the procedure of matching $\psi_{4,\mathrm{AS}}(x)$ and $\psi_{1,\mathrm{AS}}(x)$ and α by continuation $\psi_{4,\mathrm{AS}}(x)$ to the D_3, $\bar{D}_3$ sectors. They can be determined as

$$C=-i[\tilde{\psi}_{3\to\bar{3}}/(\tilde{\psi}_{\bar{3}\to 1}\tilde{\psi}_{4\to\bar{3}})]_{\mathrm{AS}}\left(1+\exp\left[\int_{K'}(\lambda q_{\mathrm{AS}}^{1/2}+\chi^{-})\,\mathrm{d}y\right]\right)^{-1}$$

$$\times\exp\left[\lambda\left(\int_{K''}+\int_{a_2}^{x_p}+\int_{K_\varepsilon}\right)q_{\mathrm{AS}}^{1/2}\,\mathrm{d}y\right]$$

and

$$\alpha=-i[(\tilde{\psi}_{4\to\bar{3}})/\tilde{\psi}_{3\to\bar{3}}]_{\mathrm{AS}} \tag{5.6}$$

where the paths K', K'' and K_ε are shown in figure 3.

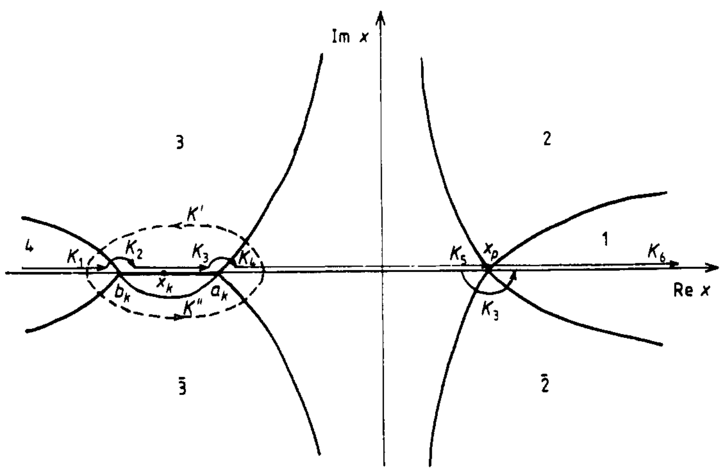

Figure 3. The Stokes graph for case 2 of § 5.

Let us note that the solution $C(\alpha\psi_{3,\mathrm{AS}}(x)+\bar{\alpha}\psi_{\bar{3},\mathrm{AS}}(x))$ continued to the part K_5 of K takes the form:

$$C(\alpha\psi_{3,\mathrm{AS}}(x)+\bar{\alpha}\psi_{\bar{3},\mathrm{AS}}(x))=(-1)^m\psi_{1,\mathrm{AS}}(x) \qquad x\in K_5 \tag{5.7}$$

where $\psi_{1,\mathrm{AS}}(x)$ can be continued to K_5 from K_6 through the point x_p which is regular for this wavefunction (see appendix 4).

From (5.6) it follows that, due to the factor $\exp(\lambda\int_{x_2'}^{x_p} q_{\mathrm{AS}}^{1/2})$, C is exponentially small when $\lambda\to+\infty$. It causes the integrals calculated on the parts K_i, $i=1,2,3,4$ (but not on K_5!), to also become exponentially small. Therefore one needs only to calculate the integrals on K_1 and K_5. But, because of (5.7), this is the same task as in the previous point, and we get simply

$$f_{mn}=f(x_p)\delta_{mn}+\mathrm{O}(\lambda^{-1}). \tag{5.8}$$

Case 2b. The conditions (5.2) defining the case are equivalent to $E_{k,m}^{(s)}=E_{p,m}^{(s)}$, for $s=0,1,\ldots,r$, and $E_{k,m}^{(r+1)}\neq E_{p,m}^{(r+1)}$. However, due to the regular behaviour of $\psi_{k\to\bar{k}}$ when $\lambda\to+\infty$ (see appendix 5) this case reduces, in fact, to the previous one and the formula (5.8) is still valid.

Case 2c. Both the points x_p and $x_k=-x_p$ are now regular for $\psi_{1,\mathrm{AS}}(x)$ and $\psi_{4,\mathrm{AS}}(x)$ respectively. Using symmetry arguments we can construct asymptotic solutions in different regions of the integration path in the following way:

$$\psi_{\mathrm{AS}}=\begin{cases}\psi_{1,\mathrm{AS}}(x) & x>x_p\\ (-1)^m\psi_{1,\mathrm{AS}}(x) & 0<x<x_p\\ \psi_{1,\mathrm{AS}}(-x) & -x_p<x<0\\ (-1)^m\psi_{1,\mathrm{AS}}(-x) & x<-x_p\end{cases} \tag{5.9}$$

where $\psi_{1,\mathrm{AS}}(x)$ is defined in the region $0<x<x_p$ by its analytic continuation from $x>x_p$ through the point x_p. It follows then that both points now contribute to the asymptotic matrix elements of f and we get

$$f_{mn}=[f(x_p)+(-1)^{m+n}f(-x_p)]\delta_{mn}+\mathrm{O}(\lambda^{-1}). \tag{5.10}$$

Case 3. The saddle-point method applied to the case collects the contributions from the well extrema of the pth well asymptotic wavefunctions as well as of the kth ones. However, the contribution of the kth well wavefunctions at the pth well bottom is exponentially small (by (5.6)) and vice versa. Therefore, the matrix element (3.8) is, in the considered case, exponentially small in comparison with the previous cases and this exponent is determined by the formula (5.6).

At the end of this section let us note that the λ asymptotic of the low energetic matrix elements also can be studied using the technique originated by Fröman and Fröman (1977). It needs, however, a generalisation to include both the case of many-well potentials and the complete asymptotic form (2.12) of the wavefunctions.

6. The Schrödinger equation in N dimensions with O(N)-symmetric potential

Let us apply the result of the previous sections to the case of the N-dimensional Schrödinger equation defined by the following Hamiltonian:

$$H=p^2/2+NV(x^2/N) \tag{6.1}$$

where $x^2 = \Sigma_{k=1}^N x_k^2$, $p^2 = \Sigma_{k=1}^N p_k^2$ with $p_k = -i\partial/\partial x_k$ (in (6.1) we have put $m = \hbar = 1$). The Hamiltonian (6.1) is, of course, $O(N)$-symmetric. The behaviour of such (and similar) systems in the limit $N \to \infty$ has recently been studied extensively by many authors (Jevicki and Papanicolaou 1980, Jevicki and Levine 1980, Bardakci 1981, Halpern 1981, Yaffe 1982, Koudinov and Smondyrev 1983, Avan 1984, Avan and de Vega 1983). Because of the scalar character of the Hamiltonian (6.1) with respect to the group $O(N)$ we can limit our investigations only to the states which form the tensor sector of the irreducible representations of the group $O(N)$. As is well known, each tensor function of $x_1, \ldots, x_N$ belonging to this sector is totally symmetric and traceless. Therefore, each wavefunction $\psi_E(x_1, \ldots, x_N)$ being a solution of the equation

$$H\psi_E = E\psi_E \tag{6.2}$$

can be represented as

$$\psi_E(x_1, \ldots, x_N) = R_{k,E}(r) T\mu_1 \ldots \mu_k(x_1, \ldots, x_N) \tag{6.3}$$

where $r^2 = x^2$ and $T\mu_1 \ldots \mu_k$ is some irreducible tensor of the $O(N)$ group. Substituting (6.3) into (6.2) we get the following radial Schrödinger equation for $R_{k,E}(r) r^{(N+2k-1)/2} \equiv Y_k(r)$:

$$Y_k'' - [2NV(r^2/N) + (N+2k-1)(N+2k-3)/(4r^2) - 2E] Y_k = 0. \tag{6.4}$$

Changing further the variables in (6.4) with the rules $r \to rN^{1/2}$, $E \to N\varepsilon$ and putting $U(r) \equiv V(r^2)$ and $\Omega_k(r) \equiv Y_k(rN^{1/2})$ we get

$$\Omega_k'' - N^2[2U_{\mathrm{Eff}} - 2\varepsilon]\Omega_k = 0 \tag{6.5}$$

where $U_{\mathrm{Eff}} = U + [1 + (2k-1)/N][1 + (2k-3)/N]/8r^2$. Although the above equation has the standard form (2.2) we cannot investigate directly the asymptotic behaviour of Ω_k and other quantities when $N \to +\infty$, using the method worked out in the preceding sections, because of the singularity of U_{Eff} at $r = 0$. A finite solution in the Fröman and Fröman form (2.3) with the γ path starting from $r = 0$ (the infinity point of U_{Eff}) does not exist. However, it is well known that such a finite solution can be constructed with the help of the following Langer change of variables in (6.5):

$$\begin{aligned} &r = \exp(x) \\ &Z_k(x) = \Omega_k(x) \exp(-x/2) \\ &-\infty < x < +\infty. \end{aligned} \tag{6.6}$$

Then, we get finally for Z_k:

$$Z_k'' - N^2 \Pi_k Z_k = 0 \tag{6.7}$$

with

$$\Pi_k(x) \equiv 2U(e^x)\, e^{2x} - 2\varepsilon\, e^{2x} + [1 + (2k-2)/N]^2/4. \tag{6.8}$$

6.1. *The low energy levels in the large-N limit*

Let us calculate first the energy levels when $N \to +\infty$. For simplicity we suppose the potential $U(r)$ to be a polynomial in r, having only one well, with the minimum at

$r = r_1 > 0$, together with $U(r_1) = 0$. Therefore, in this case the asymptotic quantisation condition (3.7) is

$$i \oint_{C'} [N\Pi_k^{1/2}(x) + \tilde{\phi}_k^-(x)]\, \mathrm{d}x = (2m+1)\pi \qquad m = 0, 1, \ldots \tag{6.9}$$

where C' is the closed path around the minimum of $\Pi(x) = \lim_{N\to+\infty} \Pi_k(x)$. Coming back in (6.9) to the variable r, we get

$$i \oint_C \{N[2U(r) + (1 + (2k-2)/N)^2/(4r^2) - 2\varepsilon]^{1/2} + \phi_k^-(r)\}\, \mathrm{d}r$$
$$= (2m+1)\pi \qquad m = 0, 1, \ldots \tag{6.10}$$

where C runs around the minimum r_0 of $U_0(r) \equiv U(r) + (8r^2)^{-1}$ and $\phi_k^-(\mathrm{e}^x) \equiv \tilde{\phi}_k^-(x)\,\mathrm{e}^{-x}$.

Making use of formulae (4.4) we obtain the following expressions for the first three terms of the asymptotic expansion of the (k, m)th energy level:

$$\varepsilon_{k,m}^{(0)} = U_0(r_0) = U(r_0) + (8r_0^2)^{-1}$$
$$\varepsilon_{k,m}^{(1)} = [(m+1/2)(U^{(2)})^{1/2} + (k-1)/(2r^2)]_{r=r_0}$$
$$\varepsilon_{k,m}^{(1)} = \tfrac{1}{2}(k-1)^2/r^2 - 1/8r^2 - \tfrac{1}{4}\{\tfrac{1}{12}[5(m+\tfrac{1}{2})^2 + \tfrac{7}{12}][2U^{(3)} - 6/r^5]^2$$
$$+ \tfrac{1}{4}[(m+\tfrac{1}{2})^2 + \tfrac{1}{4}][2U^{(4)} + 30/r^6][2U^{(2)} + 3/2r^4 + (2m+1)(k-1)$$
$$\times [U'' + 3/4r^4]^{1/2}(4U^{(3)}/r^3 + 12U''/r^4 - 3/r^8)$$
$$- [4(k-1)^2/r^6][2U'' + 3/2r^4]\}$$
$$\times [2U'' + 3/2r^4]^{-2}|_{r=r_0} \qquad k, m = 0, 1, \ldots. \tag{6.11}$$

Of course, to get the corresponding asymptotic series for the energy E given by (6.4) it is sufficient to multiply the asymptotic series for $\varepsilon_{k,m}$ by N.

Before going further let us note that, according to the general rule, the state of the lowest energy corresponds to $k = m = 0$, i.e. is non-degenerate (Glimm and Jaffe 1981).

6.2. *The large-N limit of low energy matrix elements*

Let us now investigate the behaviour of the matrix elements (3.8) with N. We only consider the case with $f(x, p)$ depending on the scalar arguments x^2 and p^2. In such a case it is sufficient to consider a monomial $f^{k,j} = x^{2k}p^{2j}$, which acts on the wavefunction (6.3) as

$$f^{kj}\psi_E = x^{2k}p^{2j}\psi_E = T\mu_1 \ldots \mu_q(-1)^j r^{2j}[r^{-N-2q+1}(\mathrm{d}/\mathrm{d}r)(r^{N+2q-1}\,\mathrm{d}/\mathrm{d}r)]^j R_{q,E}(r). \tag{6.12}$$

Using now the modified wavefunctions Ω_k we get the following expression for the matrix elements of f^{kj}:

$$f_{qm,vn}^{kj} = \delta_{qv}N^{k-j}\int_0^\infty \mathrm{d}r\, \Omega_{qm}\Sigma^{kj}\Omega_{qn}\left(\int_0^\infty \mathrm{d}r\, \Omega_{qm}^2 \int_0^\infty \mathrm{d}r\, \Omega_{qn}^2\right)^{1/2} \tag{6.13}$$

where

$$\Sigma^{kj} = (-1)^j r^{(N+2q-1)/2}$$
$$\times \{r^{2k}[r^{-N-2q+1}(\mathrm{d}/\mathrm{d}r)(r^{N+2q-1}\,\mathrm{d}/\mathrm{d}r)]^j r^{-(N+2q-1)/2}\}. \tag{6.14}$$

Evaluating the leading term for $f^{kj}_{qm,vn}$ we get in the limit $N\to+\infty$ (see appendix 7 for details)

$$f^{kj}_{qm,vn}=\delta_{qv}\delta_{mn}R_0^{2k}P_0^{2j}+O(N^{-1}) \tag{6.15}$$

where $R_0=r_0N^{1/2}$ and $P_0=(2N\varepsilon^{(0)}_{v,n}-2NU(r_0))^{1/2}=N/2R_0$ is the value of the classical momentum tangent to the sphere of radius R_0, which the classical particle moves over having the total energy $N\varepsilon^{(0)}_{v,n}$ in the external potential $NU(rN^{1/2})$. So, we get in this way the well known picture of the large-N limit of the O(N) model—the classical movement with constraints (see, for example, Yaffe (1982) and references therein). In the considered case the particle is constrained to move over the sphere of any radius R_0 and the momentum P_0 fulfilling $P_0/R_0=\frac{1}{2}r_0^{-2}$ and this limit does not depend on the state number n. This last property is caused by the settling down of the particle at the bottom of the potential well when $N\to+\infty$ so that different geometries of the states $\psi_{v,n}$ reduce to that of the sphere. For the same reason the matrix $f^{kj}_{qm,vn}$ diagonalises in the limit $N\to+\infty$, becoming a classical function of the variables R_0 and P_0. Moreover, this function appears to always be the same independently of the ordering of the operators x^2 and p^2 in the operator function $f(x^2,p^2)$, i.e. these operators commute in the limit $N\to+\infty$.

7. Summary of the results

For the quantum mechanical systems for which the corresponding Schrödinger equation can be reduced to a one-dimensional equation we have calculated the low energy levels and corresponding matrix elements. We have done it with the help of the conventional asymptotic series expansions in relevant quantities. We have shown that these calculations can be performed for potentials with arbitrary numbers of wells. We have found the following properties of the asymptotic quantisation procedure.

(i) The quantum conditions for the energies become the standard Bohr-Sommerfeld conditions independently for each well.

(ii) The dominant matrix elements are obtained for the energy levels belonging to the same quantised well.

(iii) The dominant part of the asymptotic matrix elements diagonalises and reduces to the classical value of the relevant quantity at the bottom of the quantised well.

(iv) For the quantum system in N dimensions with the O(N)-symmetric Hamiltonian this reduction leads to a classical movement with constraints.

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Appendix 1

To prove the relations (2.13) it is sufficient to consider only three cases: $\tilde{\psi}_{k\to\bar{k}}$, $\tilde{\psi}_{k\to\overline{k+1}}$ and $\tilde{\psi}_{k\to k+1}$ (see figure 1). Let us first prove the relation:

$$\tilde{\psi}_{k\to\bar{k}}=\tilde{\psi}_{\bar{k}\to k}. \tag{A1.1}$$

Taking into account (2.3) we get

$$\tilde{\psi}_{k\to\bar{k}} = 1 + \sum_{n\geq 1} (\sigma_k/\lambda)^n \int_{\infty_k}^{\infty_{\bar{k}}} dy_1 \int_{\infty_k}^{y_1} dy_2 \ldots \int_{\infty_k}^{y_{n-1}} dy_n\, \omega(y_1)\ldots\omega(y_n)$$
$$\times\{1-\exp[2\sigma_k S(y_1, y_2)]\}\ldots\{1-\exp[2\sigma_k S(y_{n-1}, y_n)]\}. \qquad \text{(A1.2)}$$

It should be noticed now that the y integrations go from ∞_k to $\infty_{\bar{k}}$ (along the integration path γ_k) through the cut (a_k, b_k). So the y slide from the upper sector k of Riemannian surface to the lower one $\bar{k}$. However, the right-hand side of (A1.2) can be rewritten in such a way to have the path γ_k starting at the very beginning from the lower sector k of the Riemann surface and ending at the upper sector $\bar{k}$. It needs only to change $q^{1/2}$ into $-q^{1/2}$ in (A1.2), i.e. to change $\omega\to-\omega$ and $S\to-S$. If we note also that $S(y_{i+1}, y_i) = -S(y_i, y_{i+1})$ then changing simultaneously the order of integrations in (A1.2) into the opposite one (i.e. using the transformations $y_i\to y_{n-i+1}$) we get

$$\tilde{\psi}_{k\to\bar{k}} = 1 + \sum_{n\geq 1} (\sigma_k/\lambda)^n \int_{\infty_{\bar{k}}}^{\infty_k} dy_1 \int_{\infty_{\bar{k}}}^{y_1} dy_2 \ldots \int_{\infty_{\bar{k}}}^{y_{n-1}} dy_n\, \omega(y_1)\ldots\omega(y_n)$$
$$\times\{1-\exp[2\sigma_k S(y_1, y_2)]\}\ldots\{1-\exp[2\sigma_k S(y_{n-1}, y_n)]\}. \qquad \text{(A1.3)}$$

Of course, the right-hand side of (A1.3) is identical with $\tilde{\psi}_{\bar{k}\to k}$. So the proof is completed.

The proof of the relations $\tilde{\psi}_{k\to\overline{k+1}} = \tilde{\psi}_{\overline{k+1}\to k}$ and $\tilde{\psi}_{k\to k+1} = \tilde{\psi}_{k+1\to k}$ is even easier and is reduced to changing the order of integrations in (A1.2) (where we should substitute k by $\overline{k+1}$ or $k+1$, respectively) by the rules $y_i\to y_{n-i+1}$. These transform S into $-S$ and interchange the lower limits of integrations in (A1.2) with the upper ones so that the total change of sign, equal to $(-1)^n$, appears in front of the multiple integral in (A1.2). In that way we get

$$\tilde{\psi}_{k\to k+1} = 1 + \sum_{n\geq 1} (-\sigma_k/\lambda)^n \int_{\infty_{k+1}}^{\infty_k} dy_1 \int_{\infty_{k+1}}^{y_1} dy_2 \ldots \int_{\infty_{k+1}}^{y_{n-1}} dy_n\, \omega(y_1)\ldots\omega(y_n)$$
$$\times(1-\exp[-2\sigma_k S(y_1, y_2)]\}\ldots\{1-\exp[-2\sigma_k S(y_{n-1}, y_n)]\} \qquad \text{(A1.4)}$$

and similar expressions for $\tilde{\psi}_{k\to\overline{k+1}}$ (by substitution of $\infty_{k+1}\to\infty_{\overline{k+1}}$ in (A1.5)). Having in mind that $\sigma_{k+1} = \sigma_{\overline{k+1}} = -\sigma_k$ it is seen that (A1.4) proves (A1.3).

A simple conclusion which follows from (A1.1) is the reality of $\tilde{\psi}_{k\to\bar{k}}$ (i.e. $\tilde{\psi}_{k\to\bar{k}} = \tilde{\psi}_{k\to\bar{k}}$) being the consequence of the assumed reality of the potential $q(x)$.

Appendix 2

We construct here the asymptotic series for $\chi(\sigma, x, E, \lambda)$ entering the formula (2.12). Inserting (2.12) into the Schrödinger equation (2.1) we get

$$\tfrac{5}{16}q_{AS}'^2 - \tfrac{1}{4}q_{AS}q_{AS}'' + (2\lambda\sigma q_{AS}^{5/2} - \tfrac{1}{2}q_{AS}q_{AS}')\chi + q_{AS}^2(\chi' + \chi^2) = 0 \qquad \text{(A2.1)}$$

where q_{AS} is defined by (2.11) and χ by

$$\chi(\sigma, x, E, \lambda) = \sum_{n\geq 0} \lambda^{-n-1}\chi_{n+2}(\sigma, x, E). \qquad \text{(A2.2)}$$

Substituting (2.11) and (A2.2) into (A2.1) and expanding $q_{AS}^{5/2}$ into the power series of λ^{-1} and equating to zero expressions at successive powers of λ^{-1} we get the following recurrent formulae for χ_{n+2}:

$$\chi_2=(\sigma/8)[q_0''(q_0-2E)-\tfrac{5}{4}q_0^2](q_0-2E)^{-5/2}$$

$$\chi_3=\tfrac{1}{2}\chi_2[(\sigma/2)q_0'(q_0-2E)^{-3/2}-5q_1(q_0-2E)^{-1}]+(\sigma/2)\{\tfrac{1}{4}[q_1''(q_0-2E)+q_0''q_1] -\tfrac{5}{8}q_0'q_1'-(q_0-2E)^2\chi_2'\}(q_0-2E)^{-5/2}$$

$$\begin{aligned}\chi_{n+4}=&\sum_{k=1}^{n+2}\chi_{n-k-4}\sum_{j=1}^{k}(-1)^j\Gamma(-\tfrac{5}{2}+j)/\{j!\Gamma(-\tfrac{5}{2})\}\sum_{\substack{p_1+\ldots+p_j=k\\ p_i\geqslant 1}}q_{p_1}\ldots q_{p_j}/(q_0)^j\\ &+(\sigma/2)\left(\frac{1}{2}\sum_{k=0}^{n+1}\chi_{n-k+3}\sum_{j=0}^{k}q_{k-j}'q_j\right.\\ &-\sum_{k=0}^{n}\sum_{p=0}^{n-k}\chi_{n-k-p+2}\chi_{p+2}\sum_{j=0}^{k}q_{k-j}q_j-\sum_{k=0}^{n+1}\chi_{n-k+3}'\sum_{j=0}^{k}q_{k-j}q_j\\ &\left.+\frac{1}{4}\sum_{\substack{k=0\\ n\geqslant 0}}^{n+2}(q_{n-k+2}^{+}q_k+\tfrac{5}{4}q_{n-k+2}'q_k')\right)q_0^{-5/2}\end{aligned}\tag{A2.3}$$

where in the formulae for χ_{n+4}, $n\geqslant 0$, q_0 means q_0-2E. For the reason that will become clear in appendix 5, let us express (A2.3) in terms of χ_{n+2} (as defined by (2.13)). We get

$$\chi_2^+=0$$

$$\chi_2^-=\tfrac{1}{8}[q_0''(q_0-2E)-\tfrac{5}{4}q_0'^2](q_0-2E)^{-5/2}$$

$$\chi_3^+=-\tfrac{1}{2}\chi_2^-(q_0-2E)^{-1/2}+\tfrac{1}{4}\chi_2^-q_0'(q_0-2E)^{-3/2}$$

$$\chi_3^-=\tfrac{1}{8}[q_1''(q_0-2E)+q_0''q_1-\tfrac{5}{2}q_0'q_1'](q_0-2E)^{-5/2}-\tfrac{5}{2}\chi_2^-q_1(q_0-2E)^{-1}$$

$$\begin{aligned}\chi_{n+4}^+=&\sum_{k=1}^{n+2}\chi_{n-k-4}^+\sum_{j=1}^{k}(-1)^j\Gamma(-\tfrac{5}{2}+j)/\{j!\Gamma(-\tfrac{5}{2})\}\sum_{\substack{p_1+\ldots+p_j=k\\ p_i\geqslant 1}}q_{p_1}\ldots q_{p_j}/(q_0)^j\\ &+\frac{1}{2}\left(\frac{1}{2}\sum_{k=0}^{n+2}\chi_{n-k+3}^-\sum_{j=0}^{k}q_{k-j}q_j'\right.\\ &-\sum_{k=0}^{n}\sum_{p=0}^{n-k}(\chi_{n-k-p+2}^+\chi_{p+2}^-+\chi_{n-k-p+2}^-\chi_{p+2}^+)\sum_{j=0}^{k}q_{k-j}q_j\\ &\left.-\sum_{k=0}^{n+1}\chi_{n-k+3}^{-\prime}\sum_{j=0}^{k}q_{k-j}q_j\right)q_0^{-5/2}\end{aligned}$$

$$\begin{aligned}\chi_{n+4}^-=&\sum_{k=1}^{n+2}\chi_{n-k-4}^-\sum_{j=1}^{k}(-1)^j\Gamma(-\tfrac{5}{2}+j)/\{j!\Gamma(-\tfrac{5}{2})\}\sum_{\substack{p_1+\ldots+p_j=k\\ p_i\geqslant 1}}q_{p_1}\ldots q_{p_j}/(q_0)^j\\ &+\frac{1}{2}\left(\frac{1}{2}\sum_{k=0}^{n+2}\chi_{n-k+3}^+\sum_{j=0}^{k}q_{k-j}q_j'\right.\\ &-\sum_{k=0}^{n}\sum_{p=0}^{n-k}(\chi_{n-k-p+2}^+\chi_{p+2}^++\chi_{n-k-p+2}^-\chi_{p+2}^-)\sum_{j=0}^{k}q_{k-j}q_j-\sum_{k=0}^{n+1}\chi_{n-k+3}^{+\prime}\sum_{j=0}^{k}q_{k-j}q_j\\ &\left.+\frac{1}{4}\sum_{k=0}^{n+2}(q_{n-k+2}''q_k+\tfrac{5}{4}q_{n-k+2}'q_k')\right)q_0^{-5/2}.\end{aligned}\tag{A2.4}$$

By mathematical induction one can infer from (A2.4) that $\chi^{\pm}_{n+2}$, $n \geq 0$, have the following analytic structure:

$$\chi^{\pm}_{n+2} = (q_0 - 2E)^{-(3n+\alpha^{\pm}_n)/2} \sum_{r,v \geq 0} f^{\pm}_{nrv}(q_0')^r (q_0 - 2E)^v \tag{A2.5}$$

where $\alpha^{+}_{2n+1} = \alpha^{-}_{2n} = 5$ and $\alpha^{+}_{2n} = \alpha^{-}_{2n+1} = 4$, and the sum in (A2.5) is a finite polynomial in the variables q_0' and $q_0 - E$ with coefficients $f^{\pm}_{nrv}$ which are finite polynomials in the variables q_0'', q_k, $k \geq 1$, and their derivatives. By detailed analysis of (A2.4) and using mathematical induction one can obtain the following properties of the polynomial on the right-hand side of (A2.4).

(i) The highest power $r^{\pm}_n$ of q_0' when $v = 0$ is

$$r^{+}_{2n+1} = r^{+}_{2n+2} = 2n+3,$$

and

$$r^{-}_{2n} = r^{-}_{2n+1} = 2n+2 \qquad n = 0, 1, \ldots. \tag{A2.6}$$

(ii) The lowest value $m^{\pm}_n$ of the sum $r + 2v$ of the powers in (A2.5) is:

$$m^{+}_{2n+1} = m^{+}_{2n+2} = 2n+3$$

and

$$m^{-}_{2n} = m^{-}_{2n+1} = 2n+2 \qquad n = 0, 1, \ldots. \tag{A2.7}$$

The above properties are important for considerations in appendix 5.

Appendix 3.

To get the quantisation condition (3.3) it is necessary to match the solutions $\psi_{1,\mathrm{AS}}$ and $\psi_{n,\mathrm{AS}}$ by analytic continuation. We can continue both the solutions to the sector D_k, for example, (see figure 1). Continuing $\psi_{1,\mathrm{AS}}$ we get

$$\psi_{1,\mathrm{AS}} = \sigma_2 \psi_{2,\mathrm{AS}} + \bar{\sigma}_2 \psi_{\bar{2},\mathrm{AS}} = \ldots = \sigma_k \psi_{k,\mathrm{AS}} + \bar{\sigma}_k \psi_{\bar{k},\mathrm{AS}} \tag{A3.1}$$

where

$$\sigma_k = i^{k-1} \exp\left[-\int_{\infty_1}^{a_2'} \chi^- \,\mathrm{d}y + \sum_{p=2}^{k-1} (-1)^p \left(\int_{C_p'} \lambda q_{\mathrm{AS}}^{1/2} \,\mathrm{d}y + \int_{C_p''} \chi^- \,\mathrm{d}y \right) + (-1)^{k+1} \int_{\infty_k}^{a_k'} \chi^- \,\mathrm{d}y + \sum_{p=2}^{k} (-1)^{p+1} \int_{K_p} \chi^- \,\mathrm{d}y \right] \prod_{q=2}^{k-1} (1 + e_q)$$

with

$$e_p = \exp\left((-1)^{p+1} \oint_{C_p} (\lambda q_{\mathrm{AS}}^{1/2} + \chi^-) \,\mathrm{d}y \right) \qquad p = 2, \ldots k-1. \tag{A3.2}$$

The integration paths C_p', C_p'' and K_p, $p = 2, \ldots, n-1$, are shown in figure 1. The closed paths C_p are defined by $C_p = C_p'' - \overline{C_p''}$, where the bar over C_p'' means the complex conjugate path.

In a completely analogous way one can continue $\psi_{n,\mathrm{AS}}$ to the sectors D_k and $\bar{D}_k$ to get

$$\psi_{n,\mathrm{AS}} = \omega_k \psi_{k,\mathrm{AS}} + \bar{\omega}_k \psi_{\bar{k},\mathrm{AS}} \tag{A3.3}$$

where

$$\omega_k = i^k \exp\left[(-1)^n \int_{\infty_n}^{b'_{n-1}} \chi^- \,dy + \sum_{p=n-1}^{k} (-1)^p \left(\int_{C'_p} \lambda q_{AS}^{1/2} \,dy + \int_{C''_p} \chi^- \,dy\right)\right.$$
$$\left. + (-1)^{k+1} \int_{\infty_k}^{a'_k} \chi^- \,dy + \sum_{p=n-1}^{k} (-1)^{p+1} \int_{K_p} \chi^- \,dy\right] \prod_{q=k+1}^{n-1} (1+e_q)e_k. \tag{A3.4}$$

To perform all the above calculations the following asymptotic representations for $\psi_{p,AS}$, $p = 1, \ldots, n-1$, and $\psi_{n,AS}$ were used:

$$\psi_{p,AS} = q_{AS}^{-1/4} \exp\left[(-1)^p \left(\int_{a_p}^{x} q_{AS}^{1/2} \,dy + \int_{\infty_p}^{x} \chi^- \,dy\right) + \int_{\infty}^{x} \chi^+ \,dy\right] \qquad p = 1, \ldots, n-1$$

and

$$\psi_{n,AS} = i^{n-1} q_{AS}^{-1/4} \exp\left[(-1)^n \left(\int_{b_{n-1}}^{x} q_{AS}^{1/2} \,dy + \int_{\infty_n}^{x} \chi^- \,dy\right) + \int_{\infty}^{x} \chi^+ \,dy\right] \tag{A3.5}$$

together with the relations:

$$\psi_{\bar{p},AS}(x) = \bar{\psi}_{p,AS}(\bar{x}) \qquad p = 1, \ldots, n. \tag{A3.6}$$

Appendix 4

We shall show below that in the case of low energy levels belonging to the pth well the solution $\psi_{1,AS}$ is regular at the point x_p (see figure 3). To do it let us note that, as follows from (A2.3), $\psi_{1,AS}(x)$ can have at most the infinite-fold pole at x_p. However, let us consider the following quantities:

$$\int_{C_p} \psi_E^2(x, \lambda) x^q \,dx \qquad q = 0, 1, \ldots \tag{A4.1}$$

where C_p is a closed path around the point x_p and $\psi_E(x, \lambda)$ is the exact solution to the Schrödinger equation corresponding to the energy E. The solution ψ_E is the holomorphic function of x at some vicinity of the point x_p since the potential $U(x)$ is holomorphic there. Therefore, each quantity (A4.1) vanishes. On the other hand, each of these quantities has a definite asymptotic behaviour when $\lambda \to +\infty$ which can be obtained by substituting $\psi(x, \lambda)$ in (A4.1) by its asymptotic values $\psi_{1,AS}(x)$. (This substitution is possible since $\psi_{1,AS}$ can be continued analytically to any point of the path C_p.) In this way we have

$$\int_{C_p} \psi_{1,AS}^2 x^q \,dx = 0 \qquad q = 0, 1, \ldots. \tag{A4.2}$$

The vanishing of (A4.2) for any $q = 0, 1, \ldots,$ is possible only if there is no pole at x_p.

Appendix 5

We shall show below that $\psi_{k,\mathrm{AS}}$ can be continued through the cut (b_k, a_k) (see figure 1) even if the points b_k, a_k collapse into x_k—the local minimum of the potential—when $\lambda \to +\infty$. It should be remembered that it can happen if the low energy levels are quantised, say in the pth well, which has the same depth as the kth one, i.e. $q_0(x_p) = q_0(x_k)$ (see conditions (5.2); note also that $k = p$ is the particular case of this situation). However, to simplify our further considerations we shall assume below that $q_0(x_k) \neq 1$ and $q_i(x) \equiv 0$ for $i \geq 1$. In such a case the following estimations hold, when $\lambda \to +\infty$:

$$q_{0,p} - 2E = (n+\tfrac{1}{2})(q''_{0,p}/2)^{1/2}\lambda^{-1} + \mathrm{O}(\lambda^{-1})$$

and

$$a_p - x_p \sim x_p - b_p = (n+\tfrac{1}{2})^{1/2}(q''_{0,p}/2)^{-1/4}\lambda^{-1/2} + \mathrm{O}(\lambda^{-1})$$

as well as

$$q_{0,k} - 2E = (n+\tfrac{1}{2})(q_{0,p}/2)^{1/2}\lambda^{-1} + \mathrm{O}(\lambda^{-1})$$

and

$$a_k - x_k \sim x_k - b_k = (n+\tfrac{1}{2})^{1/2}(q_{0,p}/q_{0,k})^{1/4}\lambda^{-1/2} + \mathrm{O}(\lambda^{-1}) \tag{A5.1}$$

where $q_{0,k} = q_0(x_k)$, $q''_{0,k} = q_0(x_k)$, etc.

The analytical continuation of $\psi_{k,\mathrm{AS}}$ through the cut (b_k, a_k) can be done along the path $U_{i=1}^{5} C_{i,k}$, as is shown in figure 4. It is seen that only the behaviour of the following quantity:

$$\int_{C_k} \chi \,\mathrm{d}x \tag{A5.2}$$

where $C_k = C_{2,k} + C_{3,k} + C_{4,k}$, is critical for this continuation.

Let us note further that the point a'_k can be chosen to lie arbitrarily close to the point x_k, independently of λ, but simultaneously keeping the point a_k between them. The latter property can be achieved by choosing λ sufficiently large. Having this in mind we can perform the integrations in (A5.2) expanding first the integrand in Taylor series around the point x_k. However, before doing it let us observe that, as follows from (A2.4), all χ^+_{n+2} with $n \geq 1$ ($\chi^+_2 \equiv 0$) only have poles at the points a_k, b_k with zero

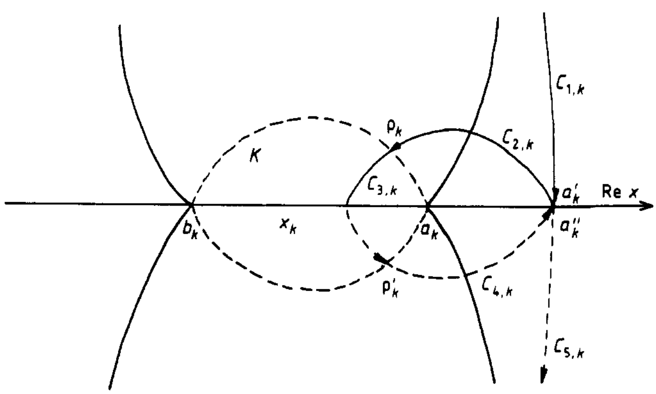

Figure 4. The integration path for the kth well.

residue. So, they do not contribute to (A5.2). Therefore, it is necessary to consider the following quantities:

$$\int_{C_k} \chi_{n+2}^-(x, E)\,dx \qquad n \geqslant 0. \tag{A5.3}$$

We first expand $(q_0 - 2E)^{-(3n+\alpha_n^-)/2+v}$ in the following way:

$$(q_0 - 2E)^{-\beta_n+v} = (q_{0,k} - 2E)^{-\beta_n+v} F(\beta_n - v, 1; 1; -(q_0 - q_{0,k})/(q_{0,k} - 2E))$$

for $x \in \operatorname{int} K$ and

$$(q_0 - 2E)^{-\beta_n+v} = (q_0 - q_{0,k})^{-\beta_n+v} F(\beta_n - v, 1; 1; -(q_{0,k} - 2E)/(q_0 - q_{0,k})) \tag{A5.4}$$

for $x \in \operatorname{ext} K$, where K is a set defined by $|q_0 - q_{0,k}| \leqslant |q_{0,k} - 2E|$, $\beta_n = (3n + \alpha_n^-)/2$ and $F(\alpha, \beta; \gamma; x)$ is the hypergeometric function.

Next, noticing that $C_{3,k} \in \operatorname{int} K$, and $C_{2,k}$, $C_{4,k} \in \operatorname{ext} K$ and $q_0 = q_{0,k} + \frac{1}{2}q''_{0,k}(x - x_k)^2 + O((x - x_k)^3)$ and keeping only the lowest powers of $\rho_k - x_k$, $\rho'_k - x_k$, $a'_k - x_k$ and $a''_k - x_k$ we get

$$\begin{aligned}
\int_{C_{3,k}} \chi_{n+2}^-\,dy &= \sum_{r,v} f_{nrv}^-(x_k)(r+1)^{-1}(q''_{0,k})^r (q_{0,k} - 2E)^{-\beta_n+v}[(\rho_k - x_k)^{r+1} \\
&\quad \times F(\beta_n - v, r/2 + \tfrac{1}{2}; r/2 + \tfrac{3}{2}; -\tfrac{1}{2}q''_{0,k}(\rho_k - x_k)^2/(q_{0,k} - 2E)) - (\rho_k \to \rho'_k)] \\
\int_{C_{2,k}} \chi_{n+2}^-\,dy &= \sum_{r,v} f_{nrv}^-(x_k)[2^{\beta_n - v}/(2\beta_n - 2s - r - 1)](q''_{0,k})^{-\beta_n+v+r}[(a'_k - x_k)^{-2\beta_n+2v+r+1} \\
&\quad \times F(\beta_n - v, \beta_n - v - r/2 - \tfrac{1}{2}; \beta_n - v - r/2 + \tfrac{1}{2}; -2(q''_{0,k})^{-1} \\
&\quad \times (q_{0,k} - 2E)/(a'_k - x_k)^2) - (a'_k \to \rho_k)] \\
\int_{C_{4,k}} \chi_{n+2}^-\,dy &= \sum_{r,v} f_{nrv}^-(x_k)[2^{\beta_n - v}/(2\beta_n - 2v - r - 1)](q''_{0,k})^{-\beta_n+v+r}[(\rho'_k - x_k)^{-2\beta_n+2v+r+1} \\
&\quad \times F(\beta_n - v, \beta_n - v - r/2 - \tfrac{1}{2}; \beta_n - v - r/2 + \tfrac{1}{2}; -2(q''_{0,k})^{-1} \\
&\quad \times (q_{0,k} - 2E)/(\rho'_k - x_k)^2) - (\rho'_k \to a''_k)].
\end{aligned} \tag{A5.5}$$

Using now the analytic continuation formulae for the hypergeometric function (Bateman 1953) we get

$$\begin{aligned}
\int_{C_k} \chi_{n+2}^-\,dy &= \sum_{r,v} f_{nrv}^-(x_k)\{\Gamma(\beta_n - v - r/2 - \tfrac{1}{2})\Gamma(r/2 + \tfrac{1}{2})/\Gamma(\beta_n - v) \\
&\quad \times (q''_{0,k})^{r/2 - 1/2}(q_{0,k} - 2E)^{-\beta_n + v + r/2 + 1/2} \\
&\quad + 2^{-r}(2\beta_n - 2v - r - 1)^{-1}(q''_{0,k})^{-\beta_n+v+r}[(a'_k - x_k)^{-2\beta_n+2v+r+1} \\
&\quad \times F(\beta_n - v, \beta_n - v - r/2 - \tfrac{1}{2}; \beta_n - v - r/2 \\
&\quad + \tfrac{1}{2}; -2(q''_{0,k})^{-1}(q_{0,k} - 2E)/(a'_k - x_k)^2) - (a'_k \to a''_k)]\}.
\end{aligned} \tag{A5.6}$$

Now let us make use of the estimations (A5.1). It is seen from (A5.3) that, when $\lambda \to +\infty$, the most singular terms should be those with the minimum values of $r + 2v$. Since the latter are given by (A2.6) we get the following leading power of λ for (A5.6):

$$\begin{aligned}
\int_{C_k} \chi_{n+2}^-\,dy &\sim \lambda^{n+1} \sum_{\substack{r,v \\ 2r+v=m_n}} f_{nrv}^-(x_k)\, n!\,\Gamma(r/2 + \tfrac{1}{2})/\Gamma(n + r/2 + \tfrac{3}{2}) \\
&\quad \times (q''_{0,k})^{r/2 - 1/2}[(m + \tfrac{1}{2})\tfrac{1}{2}(q''_{0,p})^{1/2}]^{-n-1}.
\end{aligned} \tag{A5.7}$$

Therefore, together with the power λ^{-n-1}, which stands in front of χ^-_{n+2}, (A5.6) proves the following behaviour of $\int_{x_k}^{x}\chi^-$ if continued through the cut (b_k, a_k):

$$\int_{x_k}^{x}\chi^-(x, E, \lambda)\,\mathrm{d}y = \varkappa_1^{(k)} + \sum_{n\geq 0}\lambda^{-n-1}\varkappa_{n+2}^{(k)}(x) \tag{A5.8}$$

where $\varkappa_1^{(k)}$ is a *finite* constant defined by

$$\varkappa_1^{(k)} = \lim_{\lambda\to+\infty}\int_{C_k}\chi^-\,\mathrm{d}y \tag{A5.9}$$

and given by the following series:

$$\varkappa_1^{(k)} = \sum_{n\geq 0}\sum_{\substack{r,v\\ 2r+v=m_n}} f^-_{nrv}(x_k)n!\Gamma(r/2+\tfrac{1}{2})/\Gamma(n+r/2+\tfrac{3}{2})$$
$$\times(q''_{0,k})^{r/2-1/2}[(m+\tfrac{1}{2})\tfrac{1}{2}(q''_{0,p})^{1/2}]^{-n-1}. \tag{A5.10}$$

Appendix 6

To find asymptotic expressions for the matrix elements (3.6) in the limit $\lambda\to+\infty$ we make use of the following formula:

$$\int_{-a}^{a}\exp(-\lambda h(x))f(x)\,\mathrm{d}x$$
$$\sim\sum_{r\geq 0}(\lambda/2)^{-r-1/2}\sum_{j\geq 0}^{2r}[f^{(2r-j)}(0)/(2r-j)!]\sum_{k\geq 0}^{j}[(-2)^k\Gamma(r+k+\tfrac{1}{2})/k!]$$
$$\times\sum_{\substack{p_1+\ldots+p_k=j-k\\ p_i\geq 0}}[h^{(p_1+3)}(0)\ldots h^{(p_k+3)}(0)]/[(p_1+3)!\ldots(p_k+3)!] \tag{A6.1}$$

where $h(0)=h'(0)=0$ and $h''(0)=1$, and $f(x)$ is regular at $x=0$. The above formula can be deduced, for example, from Erdelyi (1956).

Appendix 7.

We shall evaluate here the leading term for f^{kj}_{qmvn} (defined by (6.13)) in the limit $N\to+\infty$.

For the action of the operator Σ^{kj} on Ω_n we get in this limit:

$$\Sigma^{kj}\Omega_n = (-1)^j r^{(N+2q-1)/2}\{r^{2k}[r^{-N-2q+1}(\mathrm{d}/\mathrm{d}r)(r^{N+2q-1}\,\mathrm{d}/\mathrm{d}r)]^j r^{-(N+2q-1)/2}\Omega_n)$$
$$=(-1)^j N^{2j}r^{2k}[N^{-2}\,\mathrm{d}^2/\mathrm{d}r^2-[1+(2q-1)/N][1+(2q-3)/N]/4r^2]^j\Omega_n$$
$$\times(-1)^j N^{2j}r^{2k}(N^{-2}\,\mathrm{d}^2/\mathrm{d}r^2-1/4r^2)^j\Omega_n$$
$$+\mathrm{O}(N^{2j-1})(-1)^j N^{2j}r^{2k}(N^{-2}\,\mathrm{d}^2/\mathrm{d}r^2-1/4r^2)^{j-1}[(2U-2E^{(0)}_{v,n})\Omega_n]$$
$$+\mathrm{O}(N^{2j-1})(-1)^j N^{2j}r^{2k}(2U-2E^{(0)}_{v,n})^j\Omega_n+\mathrm{O}(N^{2j-1}). \tag{A7.1}$$

Therefore, for the leading term of $f^{kj}_{qm,vn}$ we obtain (see appendix 6)

$$\begin{aligned} f^{kj}_{qm,vn} &= \delta_{qv} N^{k+j} \int_0^\infty \Omega_m r^{2k} (2E^{(0)}_{v,n} - 2U)^j \Omega_n \, \mathrm{d}r / C + \mathrm{O}(N^{k+j-1}) \\ &= \delta_{qv} N^{k+j} \sum_{i \geqslant 0} (1/i!) [(2E^{(0)}_{v,n} - 2U)^j]_{r=r_0} \\ &\quad \times \int_0^\infty \Omega_m r^{2k} (r - r_0)^i \Omega_n \, \mathrm{d}r / C + \mathrm{O}(N^{k+j-1}) \\ &= \delta_{qv} \delta_{mn} N^{k+j} r_0^{2k} [(2E^{(0)}_{v,n} - 2U(r_0)]^j + \mathrm{O}(N^{k+j-1}) \end{aligned} \tag{A7.2}$$

where $C = (\int_0^\infty \Omega^2_{qm} \, \mathrm{d}r \int_0^\infty \Omega^2_{vn} \, \mathrm{d}r)^{1/2}$. The last row in (A7.2) proves (6.15).

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